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PROBLEMS OF THE CONCENTRATION OF ELASTIC STRESSES NEAR A CONICAL DEFECT†

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A method of reducing non-axisymmetric problems of the concentration of stresses near a conical crack or thin conical inclusion in the form of a conical shell to a system of one-dimensional integro-differential equations is proposed. For the case where there is torsion of the elastic medium, the exact solution of the corresponding integro-differential equation is obtained. A method of computing the quadratures (including singular quadratures) to which these problems reduce is given. There is a fundamental difference between this method and those proposed in [1, 2], in which special cases are considered in an axisymmetric formulation.

1. STATEMENT OF THE PROBLEMS AND REDUCTION OF THE LAMÉ EQUATIONS TO HARMONIC EQUATIONS

We consider an unbounded elastic medium ($0 \leqslant r < \infty, 0 \leqslant \theta \leqslant \pi, -\pi \leqslant \varphi \leqslant \pi$) with a conical defect [3, 4], by which we mean part of the surface of a circular cone

$$0 \leqslant r \leqslant R, \ \theta = \omega, \ -\pi \leqslant \varphi < \pi \tag{1.1}$$

where the displacements u_r, u_θ, u_φ and stresses $\sigma_\theta, \tau_{\theta r}, \tau_{\theta\varphi}$ have discontinuities of the first kind. A conical crack (cut), for which only the displacements have discontinuities and the stresses are continuous, is a special case. Another special case is a rigid thin conical inclusion, that is, a conical shell with mid-surface specified by relation (1.1). If this inclusion is assumed to be attached to an elastic medium, the stresses will have a discontinuity while the displacements remain continuous. All these field components will have discontinuities if the defect is a layered inclusion [3, 4].

The elastic medium is assumed to be arbitrarily loaded with a static load for which the stress and displacement fields are known

$$\sigma_\theta^0, \ \tau_{\theta r}^0, \ \tau_{\theta\varphi}^0, \ u_r^0, \ u_\theta^0, \ u_\varphi^0 \tag{1.2}$$

It is required to find the stress and displacement distributions due to this load when the defect (1.1) appears in the elastic medium.

In order to reduce the problem to integral equations, we must [3, 4] construct a discontinuous solution of the Lamé equations for defect (1.1), or in other words, a solution which satisfies Lamé's equations everywhere apart from points of the defect (1.2). The jumps of the given displacements and stresses are assigned at those points. A solution of this kind has been constructed [4] for a specific defect by solving Lamé's equations in Trefftz form.

For a conical defect it has proved to be more convenient to use a solution of Lamé's equations in Michel's form [5]. Instead of u_r, u_θ, u_φ, we introduce the functions

$$u(r,\theta,\varphi) = ru_r, \ v(r,\theta,\varphi) = r\sin\theta u_\theta, \quad w(r,\theta,\varphi) = r\sin\theta u_\varphi \tag{1.3}$$

and their Fourier transforms

$$u_n(r,\theta), v_n(r,\theta), w_n(r,\theta) = \int_{-\pi}^{\pi} \frac{u(r,\theta,\varphi), v(r,\theta,\varphi), w(r,\theta,\varphi)d\varphi}{2\pi\exp(in\varphi)}, \ n = 0, \pm 1, \pm 2, \ldots \tag{1.4}$$

Michel took the basic unknowns as the function $u(r, \theta, \varphi)$, the volume expansion $\Theta(r, \theta, \varphi)$ and radial projection of the rotation $\Omega(r, \theta, \varphi)$, with the transforms of the latter related to (1.3) by the formulae

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$$r^2\Theta_n = (ru_n)' + (\sin\theta)^{-1}v^{\cdot}_n + in(\sin\theta)^{-2}w_n \tag{1.5}$$

$$r\sin\theta\Omega_n = w^{\cdot}_n - in(\sin\theta)^{-1}v_n \tag{1.6}$$

and reduced Lamé's equations [6]

$$\begin{aligned} &2\mu_0 r\sin\theta\ \Theta^{\cdot}_n - in\Omega_n + rv^{\cdot\cdot}_n - \sin\theta r(r^{-1}u^{\cdot}_n)' = 0 \\ &2\mu_0 r^2 in\Theta_n + r\sin\theta\Omega^{\cdot}_n - inr^2(r^{-1}u_n)' + r^2 w''_n = 0 \\ &\mu_0 = (1-\mu)(1-2\mu)^{-1} \end{aligned} \tag{1.7}$$

(μ is Poisson's ratio, the third Lamé equation is omitted) to the following three separately solvable harmonic equations

$$\Delta_n\begin{pmatrix}\Theta_n\\ \Omega_n\end{pmatrix} = 0,\ \ \Delta_n f = (r^2 f')' - \nabla_n f,\ \ \nabla_n f = \frac{n^2}{\sin^2\theta}f - \frac{(\sin\theta f^{\cdot})^{\cdot}}{\sin\theta} \tag{1.8}$$

$$\Delta_n u_n = 2r^2\Theta_n - (1-2\mu)^{-1}r^3\Theta'_n \tag{1.9}$$

Here and everywhere below the prime denotes the derivative with respect to r, and the dot denotes the derivative with respect to θ.

We can represent the solution of Eq. (1.9) in the form

$$u_n = u^*_n + \tilde{u}_n \tag{1.10}$$

where u^*_n is a solution of the harmonic equation

$$\Delta_n u^*_n = 0,\ \ 0 < r < \infty,\ \ 0 < \theta < \pi \tag{1.11}$$

and u_n is a particular solution of the equation

$$\Delta_n\tilde{u}_n = 2r^2\Theta_n - (1-2\mu)^{-1}r^3\Theta'_n,\ \ 0 < r < \infty,\ \ 0 < \theta < \pi \tag{1.12}$$

If Θ_n, Ω_n, u_n have been found, we can find v_n, w_n as follows: we multiply relation (1.5) by $\sin\theta^{-1}\partial/\partial\theta\ \sin^2\theta$, and relation (1.6) by $in(\sin\theta)^{-1}$, and find the difference between the results. This leads to an equation from which to find v_n

$$-\nabla_n v_n = (\sin\theta)^{-1}\{r^2(\sin^2\theta\Theta_n)^{\cdot} + [r(\sin^2\theta u_n)^{\cdot}]'\} - inr\Omega_n \tag{1.13}$$

The other linear combination of relations (1.5) and (1.6) gives a similar equation from which to find w_n

$$-\nabla_n w_n = in[r^2\Theta_n - (ru_n)'] + r(\sin\theta)^{-1}(\sin^2\theta\Omega_n)^{\cdot} \tag{1.14}$$

2. THE CONSTRUCTION OF DISCONTINUOUS SOLUTIONS FOR THE HARMONIC EQUATIONS

In order to construct the discontinuous solution of Lamé's equations mentioned above, we first need to construct discontinuous solutions of the harmonic equations for the defect (1.1). We can use the scheme of [3, 4] for this purpose. We will apply the Mellin integral transformations to those equations on the right, using the following notation for the corresponding transforms

$$[u_{ns}, v_{ns}, w_{ns}] = \int_0^\infty \frac{[u_n(r,\theta), v_n(r,\theta), w_n(r,\theta)]}{r^{1-s}}dr$$

$$\Omega_{ns} = \int_0^\infty \Omega_n(r,\theta)r^s dr,\ \ [\Theta_{ns}, \sigma_{ns}, \tau_{rns}, \tau_{\varphi ns}] \tag{2.1}$$

$$=\int_0^\infty \frac{[\Theta_n(r,\theta),\sigma_{\theta n}(r,\theta),\tau_{rns}(r,\theta),\tau_{\varphi ns}(r,\theta)]}{r^{-1-s}}dr$$

and then the Legendre integral transformation ($P_k^m(z)$ is the adjoint Legendre function)

$$\Theta_{nsk}=\int_0^\pi \Theta_{ns}(\theta)P_k^{(n)}(\cos\theta)\sin\theta d\theta,\quad k=0,1,2,\ldots \tag{2.2}$$

by a generalized scheme [3].

As a result, for example, the Mellin transform Θ_{ns} of the discontinuous solution of the harmonic equation for Θ_n, (1.8) will be represented in the form

$$\begin{gathered}\frac{\Theta_{ns}}{\sin\omega}=\langle\dot\Theta_{ns}\rangle K_{ns}(\theta,\omega)-\langle\Theta_{ns}\rangle\frac{\partial}{\partial\omega}K_{ns}(\theta,\omega)\\ K_{ns}(\theta,\omega)=\sum_{k=|n|}^{\infty}\frac{\sigma_{kn}P_k^{|n|}(\cos\theta)P_k^{|n|}(\cos\omega)}{k(n+1)-(s+2)(s+1)},\quad 2\sigma_{kn}=(k-|n|)![(k+|n|)!]^{-1}(2k+1)\end{gathered}\tag{2.3}$$

The upper dot is used to denote the Mellin transform of the jump of the normal derivative (to the defect) of the functions $\Theta_n(r,\theta)$.

Using the inversion formula for Mellin transforms, from (2.3) we find the actual discontinuous solution

$$\Theta_n(r,\theta)=\sin\omega\int_0^R\left[\langle\dot\Theta_n\rangle\Phi_n\left(\frac{r}{\rho};\theta,\omega\right)-\langle\Theta_n\rangle\frac{\partial}{\partial\omega}\Phi_n\left(\frac{r}{\rho};\theta,\omega\right)\right]\frac{d\rho}{\rho} \tag{2.4}$$

$$\Phi_n(t;\theta,\omega)=\sum_{K=|n|}^{\infty}\frac{\sigma_{kn}}{2k+1}P_k^{|n|}(\cos\theta)P_k^{|n|}(\cos\omega)\Phi_k(t)$$

$$\Phi_k(t)=\frac{2k+1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty}\frac{t^{-(s+2)}ds}{k(k+1)-(s+2)(s+1)}=\frac{2k+1}{2\pi i}\int_{c_2-i\infty}^{c_2+i\infty}\frac{t^{-s}ds}{k(k+1)-s(s-1)}=\begin{cases}t^k, & t<1\\ t^{-k-1}, & t>1\end{cases}$$

Later we will need the limiting values of the discontinuous solution Θ_n and its derivative $\dot\Theta_n$. We can see that the following formula holds

$$\Theta_n(r,\omega\mp0)=\pm\frac{1}{2}\langle\Theta_n\rangle-\sin\omega\int_0^R\left[\langle\Theta_n\rangle\frac{\partial}{\partial\omega}\Phi_n\left(\frac{r}{\rho};\theta,\omega\right)\Big|_{\theta=\omega}-\langle\dot\Theta_n\rangle\Phi_n\left(\frac{r}{\rho};\omega,\omega\right)\right]\frac{d\rho}{\rho} \tag{2.5}$$

The formula for $\dot\Theta_n$ has the same structure, except that Φ_n in the integrand must be differentiated with respect to θ and we must put $\theta=\omega$.

The same formulae also hold for $\Omega_n(r,\theta)$ and $u_n^*(r,\theta)$, where we only need to replace $\langle\dot\Theta_n\rangle$, $\langle\Theta_n\rangle$ by $\langle\dot\Omega_n\rangle$, $\langle\Omega_n\rangle$ and $\langle\dot u_n\rangle$, $\langle u_n\rangle$ since, because the particular solution of Eq. (1.12) is continuous, we have $\langle\dot u_n\rangle=\langle\dot u_n^*\rangle$, $\langle u_n\rangle=\langle u_n^*\rangle$.

The particular solution is constructed by applying the Mellin integral transformation (2.1) and the Legendre integral transformation (2.2) to Eq. (1.12). As a result we will find the Mellin transform of the solution of Eq. (1.12) into which we substitute expression (2.3), after first replacing the transforms of the jumps by the corresponding integrals of the originals and using the orthogonality of the adjoint Legendre functions [7]. Subsequent application of the inversion formula for the Mellin transform leads to the formula

$$\begin{gathered}\tilde u_n(r,\theta)=\frac{\sin\omega}{1-2\mu}\int_0^R\left[\langle\dot\Theta_n\rangle\tilde\Phi_n\left(\frac{r}{\rho};\theta,\omega\right)-\langle\Theta_n\rangle\frac{\partial}{\partial\omega}\tilde\Phi_n\left(\frac{r}{\rho};\theta,\omega\right)\right]\rho d\rho\\ \tilde\Phi_n(t;\theta,\omega)=\sum_{k=|n|}^{\infty}\frac{\sigma_{kn}P_k^{|n|}(\cos\theta)P_k^{|n|}(\cos\omega)\Phi_k(t)}{2(2k+1)}\\ \tilde\Phi_k(t)=\begin{cases}(2k+3)^{-1}(\mu^*-k-2)t^{k+2}+(2k-1)^{-1}(k-\mu^*)t^k, & t<1\\ (2k+3)^{-1}(\mu^*+k+1)t^{-k-1}-(2k-1)^{-1}(\mu^*+k-1)t^{-k+1}, & t>1;\end{cases}\quad \mu^*=4(1-\mu)\end{gathered}\tag{2.6}$$

3. THE CONSTRUCTION OF A DISCONTINUOUS SOLUTION OF LAMÉ'S EQUATIONS

We shall assume at a defect that the jumps of the Fourier transforms of the stresses

$$\begin{aligned}
&\sigma_{\theta n}(r,\omega-0)-\sigma_{\theta n}(r,\omega+0)=\langle\sigma_{\theta n}(r,\omega)\rangle=\langle\sigma_{\theta n}\rangle\\
&\tau_{\theta rn}(r,\omega-0)-\tau_{\theta rn}(r,\omega+0)=\langle\tau_{\theta rn}(r,\omega)\rangle=\langle\tau_{rn}\rangle\\
&\tau_{\theta\varphi n}(r,\omega-0)-\tau_{\theta\varphi n}(r,\omega+0)=\langle\tau_{\theta\varphi n}(r,\omega)\rangle=\langle\tau_{\varphi n}\rangle
\end{aligned}\tag{3.1}$$

and the displacements

$$\langle u_n(r,\omega)\rangle=\langle u_n\rangle,\quad \langle v_n(r,\omega)\rangle=\langle v_n\rangle,\quad \langle w_n(r,\omega)\rangle=\langle w_n\rangle\tag{3.2}$$

are assigned.

The required discontinuous solution will have been constructed if the jumps (or Fourier transforms of the jumps) of the harmonic functions (1.8), (1.11) and (2.4) are expressed in terms of the given jumps (3.1) and (3.2).

We do this by using Hooke's law for the stress and displacement fields in a spherical system of coordinates [8].

Using the above notation and changing to Fourier transforms, we then transfer to jumps (3.1) and (3.2), and we have (G is the shear modulus)

$$\begin{aligned}
&\frac{\langle\sigma_{\theta n}\rangle}{2G}=\frac{\mu}{1-2\mu}\langle\Theta_n\rangle+\frac{\langle v_n^{\cdot}\rangle-\operatorname{ctg}\omega\langle v_n\rangle}{r^2\sin\omega}+\frac{\langle u_n\rangle}{r^2}\\
&\langle\tau_{rn}\rangle=G[r^{-2}\langle u_n^{\cdot}\rangle\operatorname{cosec}\omega r(r^{-2}\langle v_n\rangle)']\\
&\langle\tau_{\varphi n}\rangle=G(r^2\sin^2\omega)^{-1}[\sin\omega\langle w_n^{\cdot}\rangle-2\cos\omega\langle w_n\rangle+in\langle v_n\rangle]
\end{aligned}\tag{3.3}$$

Adding another two equations

$$r^2\langle\Theta_n\rangle=(r\langle u_n\rangle)'+\frac{\langle v_n^{\cdot}\rangle}{\sin\omega}+\frac{in\langle w_n\rangle}{\sin^2\omega},\quad \langle\Omega_n\rangle=\frac{\langle w_n^{\cdot}\rangle}{r\sin\omega}-\frac{in\langle v_n\rangle}{r\sin^2\omega}$$

which follow from (1.5) and (1.6) to these equations, we will find the required relations from the resulting system

$$\begin{aligned}
&\langle u_n^{\cdot}\rangle=G^{-1}r^2\langle\tau_{rn}\rangle-r^3(\sin\omega)^{-1}(r^{-2}\langle v_n\rangle)'\\
&\mu_0r^2\langle\Theta_n\rangle=\frac{r^2\langle\sigma_{\theta n}\rangle}{2G}+r\langle u_n\rangle'+\frac{\operatorname{ctg}\omega\langle v_n\rangle}{\sin\omega}+\frac{in\langle w_n\rangle}{\sin^2\omega}\\
&\langle\Omega_n\rangle=\frac{r\langle\tau_{\varphi n}\rangle}{G}+\frac{2\operatorname{ctg}\omega\langle w_n\rangle}{r\sin\omega}-\frac{2in\langle v_n\rangle}{r\sin^2\omega}
\end{aligned}\tag{3.4}$$

It only remains to obtain similar relations for $\langle\Theta_n^{\cdot}\rangle$ and $\langle\Omega_n^{\cdot}\rangle$. We find these with relations (1.7). Changing to the jumps in the latter and taking (3.5) into account, we obtain

$$\begin{aligned}
&\mu_0\sin\omega\langle\Theta_n^{\cdot}\rangle=-r\left(\frac{\langle v_n\rangle'}{r}\right)'-\left(1-\frac{n^2}{\sin^2\omega}\right)\frac{\langle v_n\rangle}{r^2}+\frac{in\operatorname{ctg}\omega\langle w_n\rangle}{r^2\sin\omega}+\\
&+(2G)^{-1}[in\langle\tau_{\varphi n}\rangle+\sin\omega(r\langle\tau_{rn}\rangle)']\\
&-r\sin\omega\langle\Omega_n^{\cdot}\rangle=in(r\langle u_n\rangle'+\langle u_n\rangle)+r^2\langle w_n\rangle''-\frac{2n^2\langle w_n\rangle}{\sin^2\omega}+\frac{2in\operatorname{ctg}\omega}{\sin\omega}\langle v_n\rangle+\frac{inr^2\langle\sigma_{\theta n}\rangle}{G}
\end{aligned}\tag{3.5}$$

To complete the construction of a discontinuous solution, it remains to solve Eqs (1.13) and (1.14). These are simple to solve if we take account of the fact that

$$\Psi_n(\theta,\tau)=\sum_{k=|n|}^{\infty}\sigma_{kn}\frac{P_k^{|n|}(\cos\theta)P_k^{|n|}(\cos\tau)}{k(k+1)}$$

is a fundamental function (fundamental solution) of the given equations. Thus the solutions of the equations can be written in the form

$$-\left\|\begin{matrix}v_n(r,\theta)\\ w_n(r,\theta)\end{matrix}\right\|=\int_0^{\pi}\Psi_n(\theta,\tau)\left\|\begin{matrix}r^2(\sin^2\tau\Theta_n)^{\cdot}+[r(\sin^2\tau u_n)^{\cdot}]'-in\sin\tau r^2\Omega_n\\ in\sin\tau[r^2\Theta_n-(ru_n)']+r(\sin^2\tau\Omega_n)^{\cdot}\end{matrix}\right\|d\tau \tag{3.6}$$

The resulting formulae have no meaning when $n = 0$ (axial symmetry). Simpler formulae can be obtained in that case. Putting $n = 0$ in (1.5) and (1.6) and integrating with respect to θ (the constant of integration turns out to be equal to zero because of the meaning of the functions v_0, w_0), we obtain

$$\left\|\begin{matrix}v_0(r,\theta)\\ w_0(r,\theta)\end{matrix}\right\|=\int_0^{\theta}\left\|\begin{matrix}r^2\Theta_0-(ru_0)'\\ r\Omega_0\end{matrix}\right\|\sin\tau d\tau \tag{3.7}$$

Formulae (3.6), (2.4), (1.10) and (3.4), (3.5) give the discontinuous solution of Lamé's equation for a conical defect (1.1) with given jumps (3.1) and (3.2).

4. REDUCTION OF THE PROBLEM TO ONE-DIMENSIONAL INTEGRO-DIFFERENTIAL EQUATIONS

It is easy to reduce any problem on stress concentration near a defect (1.1) to one-dimensional integro-differential equations, using the discontinuous solution constructed in Section 3.

In fact it is sufficient to perform the following operations for this purpose. In Fourier transforms, the required stress and displacement fields must be represented in the form of two terms

$$\sigma_{\theta n}=\sigma_{\theta n}^0+\sigma_{\theta n}^1,\quad \tau_{rn}=\tau_{rn}^0+\tau_{rn}^1,\quad \tau_{\varphi n}=\tau_{\varphi n}^0+\tau_{\varphi n}^1 \tag{4.1}$$

$$u_n=u_n^0+u_n^1,\quad v_n=v_n^0+v_n^1,\quad w_n=w_n^0+w_n^1 \tag{4.2}$$

The transforms of the components of the stress and displacement fields marked with a zero are taken from the solution (1.2), allowing for (1.3)–(1.6), while those marked with a one are taken from the formulae for the discontinuous solution of Lamé's equation given in Section 3 which, in the general case, contains six unknown jumps (3.1) and (3.2).

If the nature of the defect is sufficiently general, for example, if there is a fixed thin inclusion [3, 4], one edge (side) of which $\theta = \omega + 0$ is attached to an elastic medium, while the other $\theta = \omega - 0$ has become detached and does not interact with the medium, then there are six conditions on the defect (1.1)

$$\sigma_{\theta n}^1\big|_{\theta=\omega-0}=-\sigma_{\theta n}^0\big|_{\theta=\omega},\quad \tau_{rn}^1\big|_{\theta=\omega-0}=-\tau_{rn}^0\big|_{\theta=\omega},\quad \tau_{\varphi n}^1\big|_{\theta=\omega-0}=-\tau_{\varphi n}^0\big|_{\theta=\omega} \tag{4.3}$$

$$u_n^1\big|_{\theta=\omega+0}=-u_n^0\big|_{\theta=\omega},\quad v_n^1\big|_{\theta=\omega+0}=-v_n^0\big|_{\theta=\omega},\quad w_n^1\big|_{\theta=\omega-0}=-w_n^0\big|_{\theta=\omega} \tag{4.4}$$

Conditions (4.3) ensure that there are no stresses on the detached edge (side) of the inclusion, and conditions (4.4) ensure that the inclusion does not move. If these conditions are realized using formulae (3.6), (2.4), (1.10) and (3.4), (3.5), the problem will reduce to a system of six one-dimensional integro-differential equations (cf. [3, 4]).

We will demonstrate the corresponding operations on a defect (1.1) in the form of a crack. In that case the jumps in the stresses (3.1) will be zero, since conditions (4.3) will be satisfied on both sides, that is, both when $\theta = \omega + 0$ and when $\theta = \omega - 0$. Then the three conditions (4.3) need only be satisfied relative to the three unknown jumps of displacements (3.2), formulae (3.4) and (3.5) taking a simpler form.

We will now discuss the method in the axisymmetric case, that is, when $n = 0$. Relations (3.4) and (3.5) are even simpler in that case

$$\langle u_0\rangle = -\frac{r^3}{\sin\omega}\left(\frac{\langle v_0\rangle}{r^2}\right)', \quad \frac{\langle\Theta_0\rangle}{\mu_0^{-1}} = \frac{\langle u_0\rangle'}{r} + \frac{\operatorname{ctg}\omega\langle v_0\rangle}{r^2\sin\omega}$$

$$\langle\Omega_0\rangle = \frac{2\operatorname{ctg}\langle w_0\rangle}{r\sin\omega}, \quad \mu_0\langle\Theta_0\rangle = \frac{-r}{\sin\omega}\left(\frac{\langle v_0\rangle'}{r}\right)' - \frac{\langle v_0\rangle}{r^2\sin\omega} \tag{4.5}$$

$$\sin\omega\langle\Omega_0\rangle = -r\langle w_0\rangle''$$

Conditions (4.3) take the form

$$\sigma_\theta^1(r,\omega-0) = -\sigma_\theta^0(r,\omega), \quad \tau_{\theta r}^1(r,\omega-0) = -\tau_{\theta r}^0(r,\omega) \tag{4.6}$$

$$\tau_{\theta\varphi}^1(r,\omega-0) = -\tau_{\theta\varphi}^0(r,\omega), \quad 0\le r\le R$$

According to the formulae for the stresses, which are the same as formulae (3.3), from which the symbols for the jumps must be removed and in which ω must be replaced by θ, and also according to (3.7)

$$\frac{\sigma_\theta^1}{2G} = \mu_0\Theta_0 + \frac{[u_0-(ru_0)']\sin\theta - \operatorname{ctg}\theta v_0}{r^2\sin\theta}, \quad \frac{\tau_{\theta r}^1}{G} = \frac{u_0}{r^2} + r\left(\frac{v_0}{r^2}\right)'$$

$$\frac{r\tau_{\theta\varphi}^1}{G} = \Omega_0 - \frac{2\operatorname{ctg}\theta}{\sin\theta}\int_0^\theta \sin\tau\,\Omega_0(r,\tau)\,d\tau \tag{4.7}$$

Hence this problem splits into two problems that can be solved independently of one another: (1) the axisymmetric deformation of an elastic medium with unknown jumps $\langle u_0\rangle$ and $\langle v_0\rangle$, determined from the first two conditions of (4.6), and (2) the torsion of the elastic medium with unknown jump $\langle w_0\rangle$ determined from the last condition of (4.6).

As an example to show how optimal integro-differential equations are obtained under conditions (4.6), we will consider the problem of torsion, or the last condition of (4.6). The key point here is the choice of the unknown function. It is this which defines the equation to which the problem reduces. The optimal function in this case is the following

$$\chi(r) = \langle w_0(r,\omega)\rangle r^{-1}, \quad \operatorname{supp}\chi(r) = [0,R] \tag{4.8}$$

We will multiply both sides of the last equation from (4.6) by G^{-1} and for the given right-hand side introduce the notation

$$r\tau_{\theta\varphi}^0(r,\omega) = \tau_-(r), \quad \operatorname{supp}\tau_-(r) = [0,R]$$

Then substituting the expression $\tau_{\theta\varphi}^1$ from (4.7) into the left-hand side an taking account of formulae (2.4), (2.5) and (4.5) for $\Omega_0(r,\theta)$, using the following, easily verified relations

$$\frac{\partial^2}{\partial\rho^2}\Phi_k\left(\frac{r}{\rho}\right) = -r\frac{\partial}{\partial r}\frac{1}{\rho^2}\Phi_k^0\left(\frac{r}{\rho}\right), \quad \Phi_k(t) = \begin{cases} -(k+1)t^k, & t<1 \\ kt^{-k-1}, & t>1 \end{cases}, \quad k=0,1,2,\dots$$

$$r\frac{\partial}{\partial r}\Phi_k\left(\frac{r}{\rho}\right) = \Phi_k\left(\frac{r}{\rho}\right), \quad \Phi_k^1(t) = \begin{cases} k^{-1}t^k, & t<1 \\ -(k+1)^{-1}t^{-k-1}, & t>1 \end{cases}, \quad k=1,2,\dots$$

we obtain an integro-differential equation of Wiener–Hopf type

$$\frac{\operatorname{ctg}\omega}{\sin\omega}\chi(r) + r\frac{\partial}{\partial r}\int_0^R \frac{\chi(\rho)}{\rho}k\left(\frac{r}{\rho}\right)d\rho = -\frac{\tau_-(r)}{G}, \quad 0\le r\le R \tag{4.9}$$

Its kernel is given by the formulae

$$k(t)=k_1(t)-2\operatorname{ctg}\omega k_3(t)-2\operatorname{ctg}\omega\operatorname{cosec}\omega k_2(t)+4\operatorname{ctg}^2\omega\operatorname{cosec}\omega k_4(t)$$

$$[k_1(t),k_2(t)]=\sum_{k=0}^{\infty}\frac{\Phi_k^0(t)}{2}[|P_k(\cos\omega)|^2,\ B_k(\omega)],\quad [k_3(t),k_4(t)]=\sum_{k=1}^{\infty}\frac{\Phi_k^1(t)}{2}[A_k(\omega),C_k(\omega)] \tag{4.10}$$

$$[A_k(\omega),C_k(\omega)]=\left[P_k(\cos\omega),\ \int_0^{\omega}\sin\tau P_k(\cos\tau)\right]\frac{d}{d\omega}P_k(\cos\omega)$$

$$B_k(\omega)=P_k(\cos\omega)\int_0^{\omega}\sin\tau P_k(\cos\tau)d\tau,\quad \frac{dP_k(\cos\omega)}{d\omega}=P_k^1(\cos\omega)$$

5. THE CONSTRUCTION OF THE EXACT SOLUTION OF THE INTEGRO-DIFFERENTIAL EQUATION OF THE TORSION OF AN ELASTIC MEDIUM WITH A CONICAL CRACK

We will construct the exact solution of Eq. (4.9). Following the procedure of the factorization method (see [9], for example) we introduce the additional unknown

$$\tau_+(r)=r\tau_{\theta\varphi}^1(r,\omega-0),\quad r>R,\quad \operatorname{supp}\tau_+(r)=(R,\infty) \tag{5.1}$$

which is the required shear stress on the continuation of the conical crack (1.1). Adding (5.1) with the appropriate multiplier to the right-hand side of Eq. (4.9), we extend it over the entire axis $0\leqslant r<\infty$. Then making the replacement $r=\xi R$, $\rho=\eta R$ and carrying out a Mellin transformation we obtain a Wiener–Hopf functional equation, given on the imaginary axis

$$X^-(s)\left[\frac{\operatorname{ctg}\omega}{\sin\omega}-sK(s)\right]=\frac{T^+(s)-T^-(s)}{G} \tag{5.2}$$

where

$$[X^-(s),\ T^-(s)]=\int_0^1\xi^{s-1}[\chi(\xi R),\tau_-(\xi R)]d\xi$$

$$T^+(s)=\int_1^{\infty}\tau_+(R\xi)\xi^{s-1}d\xi,\quad K_j(s)=\int_0^{\infty}k_j(t)t^{s-1}dt,\quad j=1,2,3,4 \tag{5.3}$$

$$K(s)=K_1(s)-2\operatorname{ctg}\omega K_3(s)-2\operatorname{ctg}\omega\operatorname{cosec}\omega[K_2(s)-2\operatorname{ctg}\omega K_4(s)]$$

The minus (plus) in the subscript is, as usual, taken to denote that the function is analytic in the right (left) half-plane $\operatorname{Re}s \gtrless 0$. In order to solve Eq. (5.2), as we know [9], we must factorize the coefficient of $X^-(s)$. In order to do this, we must study the behaviour of the symbols $K_j(s)$ of the kernels $k_j(t)$ ($j=1, 2, 3, 4$) at infinity.

We will begin with the symbol $K_1(s)$ of the kernel $k_1(t)$. We separate out its principal part. Using Laplace's asymptotic formula [10]

$$P_k(\cos\omega)=2^{1/2}(\pi k\sin\omega)^{-1/2}\cos[(k+1/2)\omega-1/4\pi]+O(k^{-3/2}),\quad k\to\infty$$

we confirm the validity of the approximate formula

$$k_1(t)\approx\sum_{k=0}^{k_0}\frac{[P_k(\cos\omega)]^2}{2}\Phi_k^1(t)+\sum_{k=k_0+1}^{\infty}\frac{1+\sin(2k+1)\omega}{2\pi k\sin\omega}\Phi_k^1(t) \tag{5.4}$$

which will be more exact the larger we take the number k_0. The resulting formula shows that the principal part of the kernel $k_1(t)=\tilde{k}_1(t)+k_1^*(t)$ will be the function

$$k_1(t) = \frac{(\sin\omega)^{-1}}{2\pi}\left[\sum_{k=1}^{\infty}\begin{cases}-t^k, & t<1\\ t^{-k-1}, & t>1\end{cases} = \begin{cases}-(1-t)^{-1}, & t<1\\ (t-1)^{-1}, & t>1\end{cases}\right] \tag{5.5}$$

and its symbol will have the form

$$K_1(s) = \frac{-\operatorname{ctg}\pi s}{2\sin\omega} + K_1^*(s), \quad K_1^*(s) = \sum_{k=1}^{k_0}\left(\frac{\sin\omega\,|P_k(\cos\omega)|^2}{(\pi k)^{-1}} - 1\right)\Lambda_k(s) -$$

$$-\sum_{k=0}^{\infty}\frac{|P_k(\cos\omega)|^2}{2(k+s)} + \sum_{k=k_0+1}^{\infty}\frac{\Lambda_k(s)}{\operatorname{cosec}(2k+1)\omega}, \quad \Lambda_k(s) = \frac{(1-2s)\operatorname{cosec}\omega}{2\pi(s+k)(s-k-1)} \tag{5.6}$$

and decreases as $s \to \infty$. Proceeding in the same way as with other kernels, i.e. identifying their principal parts also, taking into account the asymptotic forms

$$A_k(\omega) = O(1), \quad B_k(\omega) = O(k^{-2}), \quad C_k(\omega) = O(k^{-1}), \quad k \to \infty$$

we can see that the remaining symbols $K_j(s)$ ($j = 2, 3, 4$) decrease as $s \to \infty$ and are given by the formulae

$$K_2(s) = \sum_{k=0}^{\infty} B_k(\omega)\Delta_k^*(s), \quad \Delta_k^*(s) = \frac{(2k+1)(1-s)}{2(s+k)(s-k-1)}$$

$$[K_3(s), K_4(s)] = -\sum_{k=1}^{\infty}\frac{\Delta_k^*(s)[A_k(\omega), C_k(\omega)]}{k(k+1)} \tag{5.7}$$

As we can see, the function to be factorized increases as $O(s)$, $s \to \infty$. We will therefore represent it in the form of the product of the function given by the formula

$$G(s) = 1 + 2\operatorname{tg}\pi s[s^{-1}\operatorname{ctg}\omega - \sin\omega K_1^*(s) + 2\cos\omega K_3(s) +$$

$$+2\operatorname{ctg}\omega K_2(s) - 4\operatorname{ctg}^2\omega K_4(s)] \tag{5.8}$$

which tends to unity as $s \to \infty$, and the function $s \operatorname{ctg} \pi s$, which has the required increase at infinity (its factorization is known, cf. [9], for example). The function (5.8), however, is factorized by the well-known formula (e.g. [11, 12])

$$G^{\pm}(s) = \exp\left[\pm\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\ln G(t)}{t-s}dt\right], \quad \operatorname{Re}s \lessgtr 0 \tag{5.9}$$

Once the coefficient of $X^-(s)$ in (5.2) has been factorized, the required functions $X^-(s)$ and $T^+(s)$ are easy to find in explicit form by the factorization method of [9, 12] and, as has been shown [9, p. 58], this is best done on Eq. (4.9) with a special right-hand side, for the case where

$$\tau_-(R\xi) = \xi^p, \quad \operatorname{Re}p > 0, \quad T_p^-(s) = (s+p)^{-1} \tag{5.10}$$

With this approach we obtain the following formula for the shear stresses on the continuation of the crack

$$\tau_{\theta\varphi}(r,\omega) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} T^-(-p)\tau_{\theta\varphi}^{(p)}(r,\omega)dp$$

$$\frac{\tau_{\theta\varphi}^{(p)}(r,\omega)}{r^{-1}} = \left(\frac{r}{R}\right)^p - \frac{1}{2\pi i}\frac{\tilde\Gamma^+(p)}{G^+(-p)}\int_{-i\infty}^{i\infty}\frac{G^+(s)}{\tilde\Gamma^+(s)}\left(\frac{r}{R}\right)^{-s}\frac{ds}{s+p}, \quad r > R \tag{5.11}$$

$$\tilde\Gamma^{\pm}(s) = \Gamma(\tfrac{1}{2} \mp s)\Gamma^{-1}(1 \mp s)$$

6. DERIVATION OF A FORMULA FOR THE STRESS INTENSITY FACTOR. A METHOD OF COMPUTING QUADRATURES

We will first establish a formula for the shear stress intensity factor

$$N_p = \lim \sqrt{2\pi(r-R)}\tau_{\theta\varphi}^{(p)}(r,\omega), \quad r \to R \tag{6.1}$$

for the special form (5.10) of loading the elastic medium. In order to take the limits, we must pick out the principal part of the second integral in (5.11), which has a radical singularity. To do this we must take into account that as $s \to \infty$ $G^+(s) - 1$ tends to zero, and so the principal part is contained in the integral

$$J^p(\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\xi^{-s}ds}{\tilde{\Gamma}^+(s)(s+p)} = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\Gamma(1-s)\xi^{-s}ds}{\Gamma(\tfrac{1}{2}-s)(s+p)}, \quad \xi = \frac{r}{R}$$

Evaluating the last integral using the theory of residues and using formula 9.131(2) of [7], we can write formula (6.1) as

$$N_p = -\frac{\tilde{\Gamma}^+(-p)}{G^+(-p)R}\lim_{r\to R}\sqrt{2\pi(r-R)}J^p(r/R) = \sqrt{\frac{2}{R}}\frac{\tilde{\Gamma}^+(-p)}{G^+(-p)} \tag{6.2}$$

For arbitrary loading, the formula for the stress intensity factor is

$$N = \lim \sqrt{2\pi(r-R)}\tau_{\theta\varphi}(r,\omega), \quad r \to R$$

Using the same argument employed to obtain formulae (5.11), we arrive at the equation

$$N = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} T^-(-p)N_p dp = \sqrt{\frac{2}{R}}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{T^-(s)\tilde{\Gamma}^+(s)}{G^+(s)}ds$$

As we see, we need to know the limiting values of the function $G^+(s)$ on the imaginary axis ($\operatorname{Re} s = 0$). We will find them from (5.9), using the Sokhotskii formula ([12], for instance). Consequently, the shear stress intensity factor for an arbitrarily loaded elastic medium is given by

$$N = \sqrt{\frac{2}{R}}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{T^-(s)\Gamma^+(s)}{\sqrt{G(s)}}\exp\left[-\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\ln G(t)}{t-s}dt\right]ds \tag{6.3}$$

We will use the following method to compute the resulting quadratures, of which one is singular. We change the variables

$$s = (\sigma+1)(\sigma-1)^{-1}, \quad t = (\tau+1)(\tau-1)^{-1} \tag{6.4}$$

which conformally maps the left half-plane of the complex variables s, t onto the unit circle, the imaginary axis becoming a circle γ of unit radius. We then apply the quadrature formula described and justified in [13]. Finally we obtain

$$N = \sqrt{\frac{2}{R}}\frac{(-1)}{2n+1}\sum_{k=0}^{2n}\frac{R(\sigma_k)}{\sigma_k - 1}\left[2\sigma_k - \frac{1+\sigma_k^{2n+1}}{\sigma_k^n}\right]\times$$

$$\times \exp\left[\frac{1-\sigma_k}{4n+2}\sum_{j=0}^{2n}\frac{g(\tau_j)}{\tau_j - \sigma_k}\left(2\tau_j - \frac{\sigma_k^{2n+1}+\tau_i^{2n+1}}{\sigma_k^n\tau_j^n}\right)\right]$$

$$g(\tau) = \ln G\left(\frac{\tau+1}{\tau-1}\right)\frac{1}{\tau-1}$$

$$R(\sigma) = T^{-}\left(\frac{\sigma+1}{\sigma-1}\right)\Gamma\left(\frac{3+\sigma}{2-2\sigma}\right)\Gamma^{-1}\left(\frac{2}{1-\sigma}\right)\left[G\left(\frac{\sigma+1}{\sigma-1}\right)\right]^{-1/2}\frac{1}{1-\sigma}$$

where the points σ_k, τ_j ($j, k = 0, 1, 2, \ldots, 2n$) divide the circle into $2n + 1$ equal parts.

REFERENCES

1. ZLATINA, I. N., Application of dual integral equations to the problem of torsion of an elastic space, weakened by a conical crack of finite dimensions. *Prikl. Mat. Mekh.*, 1972, **36**, 6, 1130–1135.
2. MARTYNENKO, M. A., The axisymmetric problem of the theory of elasticity for a space with a conical crack. *Dokl. Akad. Nauk Ukr. SSSR. Ser. A*, 1985, 5, 35–40.
3. POPOV, G. Ya., *The Concentration of Elastic Stresses near Punches, Cuts, Thin Inclusions and Reinforcements*. Nauka, Moscow, 1982.
4. POPOV, G. Ya., Problems of stress concentration in the neighbourhood of a spherical defect. *Advances in Mechanics*, 1992, **15**, 1–2, 71–110.
5. MICHEL, J. H., Some elementary distributions of stress in three dimensions. *Proc. Math. Soc. London*, 1901, **32**, 23–30.
6. LYAV, A., *Mathematical Theory of Elasticity*. Gostekhizdat, Moscow, 1935.
7. GRADSHTEIN, I. S. and RYZHIK, I. M., *Tables of Integrals, Sums, Series and Products*. Nauka, Moscow, 1971.
8. NOWACKI, W., *Theory of Elasticity*. Mir, Moscow, 1975.
9. POPOV, G. Ya., *Contact Problems for a Linearly Deformed Base*. Vishcha Shkola, Kiev, 1982.
10. SZEGŐ, G., *Orthogonal Polynomials*. Am. Math. Soc. Colloq. Publ, Vol. XXIII, New York, 1959.
11. KREIN, M. G., Integral equations on a half-line with a kernel that depends on the difference in the arguments. *Uspekhi Mat. Nauk*, 1958, **13**, 5, 3–120.
12. GAKHOV, F. D., *Boundary-value Problems*. Nauka, Moscow, 1977.
13. BELOTSERKOVSKII, S. N. and LIFANOV, I. K., *Numerical Methods in Singular Integral Equations and Their Application in Aerodynamics, the Theory of Elasticity and Electrodynamics*. Nauka, Moscow, 1985.

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